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NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

FINAL REPORT No. BR-25

Basic Mathematical Research for Electromagnetic Theory

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NEW YORK UNIVERSITY

Institute of Mathematical Sciences

Division of Electromagnetic Research

FINAL REPORT

(BR-25)

under

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Subject:

Basic Mathematical Research for Electromagnetic Theory

Morris Kline

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Director

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1. Introduction

This report is an account of the work done under Contract No. AF18(600)-367 which the Air Office of Scientific Research of the Air Research and Development Command granted to the Division of Electromagnetic Research of New York University's Institute of Mathematical Sciences. The work was begun on June 15, 1952 and terminated on June 14, 1958. The objective of the research under this contract was to explore mathematical problems which arise in the field of theoretical electromagnetics, or to anticipate mathematical needs or methodology for electromagnetic problems. The Division of Electromagnetic Research has been tackling electromagnetic problems for a number of years and has found that many investigations have been hampered by the lack of available mathematical information or of methods. These purely mathematical tools could not be freely developed under contracts emphasizing electromagnetics proper without diverting too much manpower. The Division of Electromagnetic Research was therefore happy to receive the support for the basic mathematical work which the Air Office of Scientific Research agreed to give. We wish to express our indebtedness to this office and to Dr. Merle Andrew and Mr. Robert H. Pohrer for their understanding of our needs and for the freedom of research which the contractural arrangement afforded.

Since the problems we sought to pursue arose in many different electromagnetic investigations, this account of our work is necessarily somewhat disconnected. However, the research does lie in the fields of ordinary and partial differential equations, integral equations, special functions, and asymptotic series solutions of electromagnetic problems. The individual research reports were naturally issued as any one piece of work was completed. To indicate, however, the continuity of effort in the few subjects mentioned above this account will survey the work done under those headings. References to report numbers (e.g., BR-1, etc.) indicate the precise reports in which the particular efforts are more fully described and the results established.

Appended to this report is a list of all research reports submitted under the contract and information regarding publications as of June, 1958.

2. Research in Ordinary Differential Equations

A good deal of our theoretical work has been concerned with special problems in ordinary differential equations. One major task, half of which was completed during the period of this contract, was to write a monograph on Hill's differential equation,

$$y'' + Q(x)y = 0, \qquad Q(x + \pi) = Q(x).$$

Since the research on this equation has been extensive in recent years, the purpose of the monograph was to survey and organize the existing knowledge. Part I (of two parts) has been submitted (BR-22) and contains a detailed account of the elements of the theory of Hill's equation, a review of the more sophisticated recent results, and bibliography. The report presents Floquet's theorem, a study of characteristic values of

$$y'' + (\lambda + Q(x))y = 0,$$

the results on stability of solutions, - that is, bounded solutions, - the discriminat of Hill's equation, asymptotic behavior of the characteristic values of λ, and a few facts on what is called the coexistence problem, that is, the problem of what conditions on Q permit values of λ for which two linearly independent solutions (satisfying the same homogeneous boundary conditions) can exist.

Part II of the survey will contain results on special cases of Hill's equation, particularly Lamé's and Mathieu's equation, and some applications. However, this part will appear under a new contractural arrangement.

In new work of our own on equations belonging to Hill's general type we have considered (BR-1) the question of the existence of even and odd solutions of Mathieu's equation with period π or 2π. This problem leads to the consideration of a determinant of infinite order which arises when one expresses a solution of Mathieu's equation in the form

$$y = \sum_{-\infty}^{\infty} c_n \exp(i2nx).$$

The infinite determinant called Hill's determinant, involves the c_n's. The vanishing of this determinant is necessary and sufficient for the existence of a solution of period π and a similar statement applies for solutions of period 2π. The chief result of this study is to show the relationship between Hill's determinant and the values of the solutions of Mathieu's equation at $x = \pi/2$ and $x = \pi$. The application of the theory of Mathieu's equation to the construction of transparent layers in an infinite medium is described.

In a second paper on the subject (BR-6) the results of the previous report were generalized to hold for a fairly general form of Hill's equation and some of the earlier proofs were simplified.

Several other reports were concerned with quite different problems in ordinary differential equations. In one report (BR-4) we consider the non-selfadjoint differential equation

$$u'' + q(x)u + \lambda\left[p(x)u - u'\right] = 0, \qquad u(0) = u(1) = 0.$$

The special feature of this equation is the occurrence of the term $\lambda u'$. The problem considered is whether any function can be expanded in terms of the eigenfunctions of this differential equation, a fact which is of considerable importance for applications in boundary value problems. The paper derives a condition under which a function of bounded variation can be expanded in a series of the eigenfunctions.

A rather abstract result which is applicable to ordinary differential equations (BR-16) was developed for general operators L occurring in equations such as

$$(1) \qquad Lx = a.$$

Results of interest are known for the case where the range of L is a closed set in the Hilbert space in which L is defined - notably results on existence of solutions. The present paper states conditions under which operators possess the property of having a closed range in the Hilbert space.

3. Research on Systems of Ordinary Differential Equations

Though the effect of the ionosphere on electromagnetic wave propagation has been studied for fifty years, definitive results are lacking. The reason is that the ionosphere in the presence of the earth's magnetic field presents a nonhomogeneous anisotropic medium with several parameters such as collision frequency, the gyromagnetic frequency of the electrons, and density of the electrons, each varying as a function of position and time, about which quantitative knowledge is fragmentary. Even if the parameters and consequently the dielectric tensor were known precisely the mathematical problem of predicting the behavior of a plane wave or dipole field which enters the ionosphere from below is still beyond the full powers of present-day mathematics. Awareness of this fact has led us to undertake fundamental theoretical work bearing on this problem.

On the assumption that the ionosphere is horizontally stratified so that the dielectric tensor varies in one direction only, Maxwell's equations applied to this anisotropic medium can be reduced to a system of four first-order linear ordinary differential equations in which four of the six components of E and H are the dependent variables. Hence we have undertaken to study the theory of such systems of differential equations. This theory had not been advanced very far beyond the derivation of existence and uniqueness theorems. Even the analogs of some of the many artifices which allow one to simplify or to standardize second order ordinary differential equations for one unknown function have hardly been considered. From the algebraic point of view, the most obvious line of approach seems to be to attempt to generalize the exponential form of the solution of a linear differential equation for a single unknown function. (Problems related to this one also play a role in the quantum theory of fields.) The purely algebraic aspects of the investigations to be described apply also to linear operators in Hilbert space.

We write (BR-3) a system of n homogeneous linear differential equations for n unknown functions in the form

$$\frac{dY(t)}{dt} = A(t)Y(t), \tag{1}$$

where A(t) is an n by n matrix the elements of which are functions of the real variable t and where Y(t) is an unknown n by n matrix the column vectors of which form a set of linearly independent solutions of the system of linear differential equations

$$\frac{dy_\nu}{dt} = \sum_{\mu=1}^{n} a_{\nu,\mu}(t)\dot{y}_\mu, \qquad \nu = 1,2,\ldots,n. \tag{2}$$

In (2) the $a_{\nu,\mu}$ are the elements of A and $y_1,\ldots,y_n$ are scalar functions of t.

We now try to find an n by n matrix $\Omega(t)$ such that

$$Y = \exp \Omega(t) = I + \Omega + \frac{\Omega^2}{2!} + \cdots \tag{3}$$

is a solution of (1) satisfying Y(0) = I, where I denotes the identity matrix. For sufficiently small values of t, $\Omega(t)$ is given by an infinite series the first terms of which can be determined explicitly, viz:

$$
\begin{aligned}
\Omega = \int_0^t A(\tau)d\tau &+ \frac{1}{2}\int_0^t \left[A(\tau), \int_0^\tau A(\sigma)d\sigma\right]d\tau \\
&+ \frac{1}{4}\int_0^t \left[A(\tau), \int_0^\tau \left[A(\sigma), \int_0^\sigma A(\rho)d\rho\right]d\sigma\right]d\tau \qquad (4) \\
&+ \frac{1}{12}\int_0^t \left[\left[A(\tau), \int_0^\tau A(\sigma)d\sigma\right], \int_0^\tau A(\sigma)d\sigma\right]d\tau + \cdots
\end{aligned}
$$

In (4) the square brackets [ ] denote Lie multiplication; that is, for any two n x n matrices L and M, $[L,M] = LM - ML$. The right-hand side of (4) has the following property: If iA is an hermitian matrix then iX is again hermitian for every term X on the right-hand side of (4). Formula (4) is the continuous analog of the Baker-Hausdorff formula (i.e., of the addition theorem of the exponential function in the case of noncommutative variables).

The range of validity of (4) is a finite one in most cases. Sufficient conditions for the validity of this formula for all values of t have been derived. A case in which (4) is certainly valid for all values of t arises if the series on the right-hand side terminates. In this case, the formula provides a solution of (1) in finite terms (quadratures, exponential functions and algebraic operations), because exp Ω can be written as follows:

$$
\exp \Omega = \sum_{\nu=1}^{n} \exp\left[\lambda_\nu \frac{P_\nu(\Omega)}{P'(\lambda_\nu)}\right], \qquad (5)
$$

where the λ_ν are the eigenvalues of Ω and

$$
P_\nu(\lambda) = \frac{P(\lambda)}{\lambda - \lambda_0} \qquad (6)
$$

is a polynomial in λ which is obtained from the characteristic equation

$P(\lambda) = 0$ for Ω by (6).

We have also investigated the case where

$$\Omega(t) = \int_0^t A(\tau)d\tau \tag{7}$$

or, equivalently,

$$\left[A(t), \int_0^t A(\tau)d\tau\right] = 0 \tag{8}$$

and have found (BR-10) that (8) may hold even though it is impossible to find a constant matrix T such that $T^{-1}A(t)T$ is triangular for all t. This shows that (8) is a nontrivial test for the existence of solutions in finite terms. More general criteria for solution of the system in finite terms have also been derived.

A further result (BR-17) which tells us when the series for Ω given by (4) is finite, and therefore that the solution Y(t) can be expressed in finite terms, states that when A(t) is an analytic 2 x 2 matrix then either

$$\Omega = \int_0^t A\,dt$$

or Ω consists of infinitely many terms.

4. Research in Partial Differential Equations

Several of our papers have dealt with portions of the theory of partial differential equations which have a bearing on electromagnetic problems. In our report number BR-12 we have formulated abstractly the method of separation of variables, which is widely used in all of mathematical physics. The formulation uses the notion of direct products of operators in Hilbert space. The advantage of the formulation is that it shows more clearly the possible boundary conditions for which the method of separation of variables is applicable.

A second paper (BR-11) under the general heading of partial differential equations considers the abstract equation

$$\frac{du(t)}{dt} = A(t)u(t) + f(t)$$

wherein $A(t)$ is a non-bounded linear operator and the functions $u(t)$ and $f(t)$ are functions in a Banach space. This abstract formulation covers the Schrödinger equation of quantum mechanics, linear partial differential equations of parabolic type, and some types of hyperbolic partial differential equations. The paper considers existence and uniqueness theorems for given initial conditions.

A third paper under this subject of partial differential equations presents a uniqueness for the reduced wave equation or the elliptic partial differential equation

$$\Delta u + k^2 u = 0,$$

where k^2 is bounded but not necessarily analytic. The paper shows that the solutions vanish identically if they vanish at a point more strongly than any power of the distance to that point. An analogous theorem is proved for the equation $\Delta u = F(x,u)$ provided F satisfies a Lipschitz condition.

A fourth paper (BR-2) discusses the range of applicability of the image method for second-order linear partial differential equations. The image method is used to determine a Green's function for a region of space bounded by planes in terms of the corresponding Green's function for the same operator defined over all space. The equations, regions, and boundary conditions to which the method is applicable are determined.

5. Research on Integral Equations

Under the subject of integral equations one paper (BR-15) considers a new method of solving the Fredholm integral equation which occurs in problems of potential theory and diffraction of electromagnetic waves. The method is closely related to the Neumann series method in that it is also an iterative process and represents the solution as an infinite series. However, the series obtained by this new method is always convergent and monotonic. By the latter term we mean that each step of the iterative process definitely gives a better approximation to the exact solution.

As just noted, the Fredholm integral equation occurs in diffraction problems. The problem of diffraction by a circular aperture can be reduced (BR-20) to solving the integral equation

$$-\frac{iay}{\sqrt{\pi}} = \mathcal{E}(y) + \frac{2\alpha}{\pi}\int_0^1 G(x,y,\alpha)\,\mathcal{E}(x)dx$$

where the kernel $G(x,y,\alpha)$ is given by

$$G(x,y,\alpha) = -\frac{\pi}{4}\int_{|x-y|}^{x+y} \frac{J_1(\alpha\tau) + i\,H_1(\alpha\tau)}{\tau}\,d\tau$$

where J_1 is a Bessel function, H_1 is a Struve function, and $\alpha = ka$. The quantity a is the radius of the aperture, and k is $2\pi/\lambda$. This paper studies the asymptotic solution of the above integral equation for large α. For $\alpha = \infty$ the integral equation degenerates into an integral equation of the first kind and the solution of this equation is evaluated asymptotically. At the same time we obtain a solution of the original equation for large α.

In our research report BR-13 we show how the Wiener-Hopf integral equation can be solved in a class of problems without resorting to the factorization process and function-theoretic techniques usually employed.

One-part boundary value problems are in principle treated by separation of variables and the reduction to a Sturm-Liouville problem for the resulting ordinary differential equations. On the other hand, two-part boundary value problems are reducible to one-part problems in principle by means of the function theoretic technique used in the Wiener-Hopf integral equation method. Hence it seemed desirable to provide an abstract general theory of two-part problems in terms of the related one-part problems. This has proved possible in the case where the spectrum of the one-part problem is discrete. This analysis is best described in terms of the problem of a bifurcated waveguide in which the medium is stratified in the direction perpendicular to the bifurcation. We suppose that the problem posed by the bifurcated guide with the bifurcation extending to infinity in both directions is solved and that the unbifurcated infinite guide problem is likewise solved. These separate problems are, of course, one-part problems and reduce to one-dimensional Sturm-Liouville problems. The original two-part problem can then be solved in terms of the solutions of the separate one-part problems. The central idea is to construct directly an integral representation of the solution, whose residue expansion has two different forms which correspond to the solution for the two regions. The basic feature of the integral is the presence of certain infinite products in the denominator of the integrand of the form

$$\pi(1 - \frac{\lambda}{\lambda_n})e^{\lambda/\lambda_n}, \qquad \pi(1 - \frac{\lambda}{\mu_n})e^{\lambda/\mu_n},$$

where λ_n, μ_n are the eigenvalues for the separate problems and λ is the integration variable. These products are a generalization of the gamma function in the same way that the solution of the Sturm-Liouville problem is a generalization of the sine or cosine functions. The products are related to a factorization of the entire functions in λ which are solutions of the Sturm-Liouville equations in the same manner as the gamma functions are related to the factorization of sin πz in accordance with the identity $\sin \pi z = \pi / \Gamma(z)\Gamma(1-z)$.

6. Research on the Theory of Special Functions

Several of our papers have been devoted to the theory of special functions. Two of these (BR-18 and BR-19) are concerned with what are called addition theorems. In such theorems a function specified with respect to one coordinate system is re-expressed in terms of the same function (or class of functions) which is defined with reference to another coordinate system related by translation or rotation to the original one. Addition theorems for Bessel and Legendre functions, spherical harmonics, Mathieu functions and other special functions are known. In the first of the above-mentioned reports we give addition theorems for the functions which arise from the use of parabolic coordinates and in terms of which the wave equation $\Delta u + k^2u = 0$ is separable. These functions are also called functions of the paraboloid of revolution because the parabolic coordinate system is adapted to the treatment of a surface such as a paraboloid of revolution. Each of the several addition theorems given in this paper is adapted to a particular transformation of coordinates.

In BR-19, similar results are obtained for functions of the parabolic cylinder which play the analogous role in a parabolic cylinder coordinate system to what the functions of the paraboloid of revolution play in parabolic coordinates. In this report the parabolic cylinder functions are applied to the following problem of diffraction theory. A line source is located along the axis of a perfectly conducting parabolic cylinder and the field reflected by the cylinder is expressed by an integral involving the parabolic cylinder functions. By utilizing asymptotic expressions for these functions the integral is evaluated asymptotically for large k, where $k = 2\pi/\lambda$.

In our report, BR-23, we show that the special functions called Whittaker functions and another class of special functions called E-functions are each expansible in terms of an infinite series of functions of the same class. A number of such expansions are given. The paper also shows that certain products of Whittaker functions and of E-functions are expressible as infinite series of the respective functions.

In our report, BR-24, an integral of a product of Whittaker functions is expressed as a Whittaker function and the same is shown for the E-function.

7. Theory of Asymptotic Series Solutions and Asymptotic Evaluation of Integrals

Further progress was made in our theoretical work concerned with obtaining asymptotic series solutions of Maxwell's equations. In earlier papers (done under other auspices) we presented a theory which showed how to derive asymptotic series solutions of the form

$$(1) \qquad \sum_{n=0}^{\infty} \frac{A_n}{(i\omega)^n}$$

where the A_n are functions of x,y and z and ω is the circular frequency of the source giving rise to the electromagnetic field represented asymptotically by expression (1). The work done under the present contract (BR-14) gives a theoretical basis for asymptotic series of the form

$$(2) \qquad \sum_{n=0}^{\infty} \frac{A_n}{(i\omega)^{n+1/2}} \; .$$

The latter form is necessary in problems wherein diffraction by edges of bodies occurs.

The entire theory of asymptotic series solutions of Maxwell's equations is of importance not only because it provides methods of obtaining such series solutions where exact solutions are not obtainable but also because the theory links geometrical optics and electromagnetic theory. In expression (1) when ω is infinite, all terms but the first one vanish. This first term, the 'limit' of the expression (1) as ω becomes infinite is the usual geometrical optics solution, which is of course only an approximate representation of electromagnetic fields.

Further work on the theory of asymptotic series solutions is contained in our report, BR-21. In other papers on this subject the theory was developed by employing the relationship between the time-dependent Maxwell's equations and the time-independent equations, or, as we have sometimes described it, the relation between pulse solution and time harmonic solution. The present paper gives conditions under which an asymptotic series solution of the form (1) above holds for the reduced wave equation

$$\Delta u + k^2(x)u = 0.$$

Another study, on the asymptotic evaluation of integrals, (BR-9), presents a new method for obtaining an asymptotic representation for integrals of the form

$$\int_0^\infty e^{-px} x^{c-1} f(x)dx$$

when p is large. It is shown that if f(x) satisfies certain conditions this representation is also convergent. Numerical calculations seem to show that the first term of the representation gives a close approximation to the value of the integral for a wide range of values of p.

8. Miscellaneous Reports

Our investigations under this contract resulted in two papers which do not fall under any of the main categories but which were regarded as worthwhile. In our report number BR-7, a number of formulas are derived for the coefficients of capacitance of a domain which consists of the entire plane minus any finite number of collinear slits. It is shown that any of a broad class of domains possessing a certain symmetry property is conformally equivalent to such a 'slit-domain', and that the problem of determining the conformal correspondence is essentially one of mapping simply-connected domains rather than multiply-connected ones. As an illustration, the coefficients of capacitance of a 'bi-filar shielded cable' are computed.

The second of these independent papers (BR-8) presents a Fourier transform theorem for matrices. A Fourier theorem is proved which establishes a one-to-one correspondence between n x n matrices depending on a real parameter and n x n matrices depending on the elements of a variable hermitian matrix and satisfying certain differential equations. The analogue of the Plancherel formula is shown to be true for this Fourier theorem.

Complete list of reports and publications

BR-1	W. Magnus	Infinite determinants in the theory of Mathieu's and Hill's equations Math. Rev., 14, 1087 (1953). Math. Rev., 17, 848 (1956).	Feb., 1953
BR-2	J.B. Keller	The scope of the image method Published - Comm. Pure and Appl. Math., 6, 505 (1953). Math. Rev., 14, 877 (1953). Math. Rev., 15, 321 (1954).	Apr., 1953
BR-3	W. Magnus	Algebraic aspects in the theory of systems of linear differential equations Published - Comm. Pure and Appl. Math., 649 (1954). Math. Rev., 15, 97 (1954) Math. Rev., 16, 790 (1955).	June, 1953
BR-4	B. Friedman and L.I.Mishoe	Eigenfunction expansions associated with a non-self-adjoint differential equation Published - Pac. J. Math., 6, 249 (1956). Math. Rev., 15, 796 (1954). Math. Rev., 18, 129 (1957).	Jan., 1954
BR-5	C. Muller	The behavior of the solutions of $\triangle U=F(x,U)$ in the neighborhood of a point Published - Comm. Pure and Appl. Math., 7, 505 (1954). Math. Rev., 15, 711 (1954). Math. Rev., 16, 42 (1955).	Mar., 1954
BR-6	W. Magnus	Infinite determinants associated with Hill's equation Published - Pac. J. Math., 5, Suppl. 2, 941 (1955). Math. Rev., 16, 261 (1955).	June, 1954
BR-7	B. Epstein	Determination of coefficients of capacitance of regions bounded by collinear slits and of related regions Published - Quart. of Appl. Math., 14, 125 (1956). Math. Rev., 16, 202 (1955).	Aug., 1954
BR-8	W. Magnus	A Fourier theorem for matrices Published - Proc. Amer. Math. Soc., 6, 880 (1955) Math. Rev., 16, 785 (1955).	Oct., 1954

BR-9	B. Friedman and J.Franklin	A convergent asymptotic representation for integrals Published - Proc. Camb. Phil. Soc., 53, 612 (1957). Math. Rev., 16, 815 (1955).	Nov., 1954
BR-10	H.J. Hellman	Lie algebras arising from systems of linear differential equations	Mar., 1955
BR-11	T. Kato	Linear differential equations in Banach space Published - Proc. Conf. on Differential Eqs., Berkeley, California, Comm. Pure and Appl. Math., 9, 479 (1956). Math. Rev., 17, 64 (1956).	June, 1955
BR-12	B. Friedman	An abstract formulation of the method of separation of variables Published - Proc. of Symp. on Diff. Eqs., Univ. of Maryland, 1955, pp. 209-226 Math. Rev., 17, 178 (1956).	July, 1955
BR-13	S.N.Karp	An application of Sturm-Liouville theory to a class of two-part boundary-value problems Published - Proc. Camb. Phil. Soc. of London, 53, 368 (1957). Math. Rev., 17, 160 (1956).	Aug., 1955
BR-14	M. Kline	Asymptotic solutions of Maxwell's equations involving fractional powers of frequency Published - Comm. Pure and Appl. Math., 8, 595 (1955). Math. Rev., 17, 747 (1956).	Oct., 1955
BR-15	C. Müller	A new method for solving Fredholm integral equations Published - Comm. Pure and Appl. Math., 8, 635 (1955). Math. Rev., 17, 1215 (1956).	Nov., 1955
BR-16	B. Friedman	Operators with a closed range Published - Comm. Pure and Appl. Math., 8, 539 (1955). Math. Rev., 17, 1229 (1956).	Dec., 1955
BR-17	A. Shenitzer (AD 86595)	Exponential solutions of second-order systems Math. Rev., 17, 1086 (1956).	Apr., 1956

BR-18	H. Hochstadt (AD 87063)	Addition theorems for the functions of the paraboloid of revolution Published - Pac. Jrnl. of Math., 7, 1365 (1957). Math. Rev., 17, 1084 (1956).	May, 1956
BR-19	D.I.Epstein (AD 88990)	On the functions of the parabolic cylinder Math. Rev., 18, 30 (1957).	June, 1956
BR-20	I. Epstein (AD 110302)	On a Fredholm equation in diffraction theory Math. Rev., 18, 848 (1957).	Oct., 1956
BR-21	W. Miranker (AD 120486)	The asymptotic theory of solutions of $\triangle u + k^2u = 0$. Published - Part I, J. Math. and Mech.; 6, 847 (1957). Part II, Comm. Pure Appl. Math.; 10, 491 (1957). Part III, Annals of Math.; 67, 72 (1958). Part IV, Archiv Rat. Mech. and Anal.; 1, 139 (1957).	Apr., 1957
Br-22	W. Magnus and A. Shenitzer (AD 132362)	Hill's equation. Part I: General Theory	June, 1957
BR-23	F.M. Ragab (AD 132498)	Expansions for products of two Whittaker functions	July, 1957
BR-24	F.M.Ragab (Ad 136570)	Some formulas for the products of E-functions and Whittaker functions.	Sept., 1957

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